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Inverse Determination of Thermal Conductivity for One-Dimensional Problems

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Two finite difference procedures are presented for the inverse determination of the thermal conductivity in a one-dimensional heat conduction domain. The thermal conductivity is reconstructed from the inverse analysis based on the assumption that the temperature measurements are either available continuously over the entire domain or at discrete grid points. The convergence and stability of the computational algorithms are investigated. It is concluded that both procedures are first-order accurate methods. A comparison of the exact thermal conductivity with the one estimated was made to confirm the validity of the numerical procedures. The close agreement between the two results confirms that the proposed finite difference techniques are effective procedures for the inverse determination of thermal conductivity in a one-dimensional heat conduction domain. The methods are applicable for linear and nonlinear spatially- as well as temperature-dependent thermal conductivities. Additionally, the special feature of the present techniques is that a priori knowledge of the functional form for the thermal conductivity is not mandatory.

Nomenclature

C = arbitrary constant
F = function defined by Eq. (10)
f = function defined at the boundary
G = function defined by Eq. (10)
g = heat generation, W/m^3
k = thermal conductivity, W/m-°C
k_M = initial thermal conductivity calculated at $\partial T/\partial x = 0$ in the discrete formulation, W/m-°C
k_0 = initial thermal conductivity calculated at $\partial T/\partial x = 0$ in the continuous formulation, W/m-°C
q = heat flux, W/m^2
T = temperature, °C
t = time, s
$\hat{t}$ = selected time in the continuous formulation, s
t_{i_0} = selected time in the discrete formulation, s
x = spatial coordinate, m
x_M = spatial coordinate where $\partial T/\partial x = 0$ in the discrete formulation, m
x_0 = spatial coordinate where $\partial T/\partial x = 0$ in the continuous formulation, m

Subscripts
i, j, k = indices

Superscript
$\tilde{}$ = approximated value

Introduction

WIDE attention has been devoted to materials research in the past decade due to technological advancement. Advanced materials with low weight or the ability to withstand high temperatures are actively being developed in the nuclear, electronics, and aerospace industries. The determination of material thermophysical properties for these materials is essential in many thermal management system analyses. An increasing effort has been devoted to expand our knowledge on material properties.[1–8] Specifically, an accurate prediction of thermal conductivity is imperative to achieve an optimal thermal control system.

Inverse determination of the thermal conductivity from measured temperature data has been the topic of research by many investigators.[4–8] Most of these studies employed the least-squares method to estimate the thermal conductivity in heat conduction problems. For more details on this technique, readers should refer to Refs. 9–12.

Previously, most studies assumed that the thermal conductivity is only a function of the spatial coordinate.[13–17] However, thermal conductivities are temperature-dependent quantities in most practical engineering applications.[18] Alifanov and Mikhailov[19] developed a solution to the nonlinear inverse thermal conductivity problem by applying the conjugate gradient method. In a recent study, Tervola[20] presented a numerical method to determine the nonlinear thermal conductivity of a homogenous material from measured temperature profiles. The problem was formulated as an optimization problem where the heat conduction equation was written as a constraint. It was then solved with the Davidon–Fletcher–Powell method.[21,22] Additionally, the heat equation was solved iteratively by a finite element technique with the predictor-corrector method. More recently, Ouyang[23] performed a parameter estimation of the thermal conductivities by using the least-squares technique coupled with a finite element formulation. Both of these studies assumed that thermal conductivity is a function of temperature.

The aim of this article is to study the feasibility of using finite difference techniques to determine the thermal conductivity in a heat conduction domain. The thermal conductivity is assumed to be either a function of the spatial coordinate or temperature. In addition, it also assumes that the temperature measurement is known everywhere or at discrete grid points in the defined domain. In the past, most of the studies employed an optimization technique to obtain a least-square approximation of the conductivity function. In this study, two efficient finite difference procedures are developed to discretize the heat conduction equation. This converts the governing partial differential equation into two initial value problems. As a result, the conductivity function can be recovered by solving those two initial value problems. The ad-

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vantage of this approach is that no prior information is required on the functional form for the unknown thermal conductivity.

Several heat conduction problems have been tested with the techniques for spatially and temperature-dependent thermal conductivities. The parameter estimation problem can be either linear or nonlinear. The estimated thermal conductivities are verified by comparing with the exact forms to confirm the validity of the methods. Furthermore, the order of accuracy of the two numerical procedures is discussed. The convergence and stability of the numerical procedures are also addressed.

One-Dimensional Heat Conduction Equation

To illustrate the basic concepts associated with the proposed finite difference procedures for the inverse determination of the thermal conductivity for a heat-conduction system, a one-dimensional ($0 \le x \le 1$), time-dependent nonhomogeneous problem with heat generation is studied in this article. Figure 1 depicts the region of the problem under consideration. The medium is initially at a temperature $f_m(x)$. For times $t > 0$, the boundaries at $x = 0$ and $x = 1$ are subjected to a set of boundary conditions (B.C.) of the region R, where $R = \{(x, t)\colon 0 < x < 1, t > 0\}$. Everything outside of the region is assumed to be at zero temperature. In addition, the product of the material density and heat capacity is considered as unity. The general one-dimensional heat-conduction equation can be stated as

$$\frac{\partial T(x,t)}{\partial t} - \frac{\partial}{\partial x}\left[k(x,t)\frac{\partial T(x,t)}{\partial x}\right] = g(x,t)$$

$$\text{in } 0 < x < 1, \qquad t > 0 \tag{1}$$

Subject to the following initial condition (I.C.):

$$T(x,0) = f_m(x) \quad \text{for} \quad 0 \le x \le 1 \tag{2}$$

and general boundary conditions:

Boundary condition of the first kind

$$T(x,t) = f_i(t) \quad \text{at} \quad x = 0 \quad \text{or} \quad x = 1 \quad \text{for} \quad t > 0 \tag{3a}$$

The temperature is prescribed along the boundary surface, which is a function of time for a general case.

Boundary condition of the second kind

$$\frac{\partial T(x,t)}{\partial x} = f_j(t) \quad \text{at} \quad x = 0 \quad \text{or} \quad x = 1 \quad \text{for} \quad t > 0 \tag{3b}$$

This boundary condition prescribes the applied heat flux at the boundary surface.

Boundary condition of the third kind

$$k(x,t)\frac{\partial T(x,t)}{\partial x} + T(x,t) = f_k(t)$$

$$\text{at} \quad x = 0 \quad \text{or} \quad x = 1 \quad \text{for} \quad t > 0 \tag{3c}$$

This boundary condition specifies heat dissipation by convection on a surface to a surrounding environment at zero temperature that varies with time along the surface.

The main purpose of this study is to determine the conductivity, $k(x, t)$, at any point within the domain $R = \{(x, t)\colon 0 < x < 1, t > 0\}$, with the assumption that the temperature $T(x, t)$ is known over the entire domain or at discrete grid points.

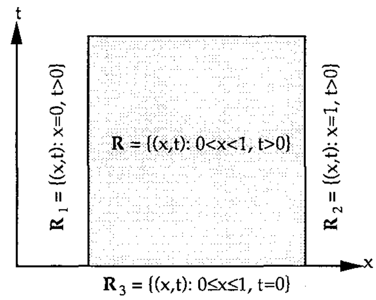

Fig. 1 Region of one-dimensional heat-conduction problem.

Inverse Determination of Thermal Conductivity

In this section, a brief discussion of the requirements that lead to a unique solution of the inverse heat conduction problem is first addressed. A numerical procedure based on the assumption that the temperature $T(x, t)$ is known over the entire domain is then formulated for the determination of the thermal conductivity. A second finite difference procedure is also presented for the recovery of the thermal conductivity based on the knowledge of the temperature $T(x, t)$ at discrete grid points.

Necessary Condition for the Uniqueness of Thermal Conductivity

Throughout this section, the temperature $T(x, t)$ is assumed known over the entire domain. As a result, derivatives of the temperature, $\partial T/\partial x$, $\partial^2 T/\partial x^2$, and $\partial T/\partial t$ can also be calculated based on the available temperature profile. The general one-dimensional heat conduction Eq. (1) can be rewritten as

$$\frac{\partial}{\partial x}\left[k(x,t)\frac{\partial T(x,t)}{\partial x}\right] = \frac{\partial T(x,t)}{\partial t} - g(x,t) \tag{4}$$

Consider the above partial differential equation at $t = \hat{t}$, the nonhomogeneous ordinary differential equation takes the form:

$$\frac{\partial}{\partial x}\left[k(x,\hat{t})\frac{\partial T(x,\hat{t})}{\partial x}\right] = \frac{\partial T(x,\hat{t})}{\partial t} - g(x,\hat{t}) \tag{5}$$

The general solution of the nonhomogeneous ordinary differential equation is the sum of the particular solution of the nonhomogeneous equation and the general solution of the homogeneous equation. Hence, let us focus only on the homogeneous case, i.e.,

$$\frac{\partial}{\partial x}\left[k(x,\hat{t})\frac{\partial T(x,\hat{t})}{\partial x}\right] = 0 \tag{6}$$

This implies

$$k(x,\hat{t})\frac{\partial T(x,\hat{t})}{\partial x} = C \tag{7}$$

where C is an arbitrary constant. If the term $\partial T(x, \hat{t})/\partial x$ is nonzero over the entire interval $[0, 1]$, one can rewrite Eq. (7) as

$$k(x,\hat{t}) = C \Big/ \frac{\partial T(x,\hat{t})}{\partial x} \tag{8}$$

Since C is an arbitrary constant, this implies that there are infinitely many solutions for the homogeneous Eq. (6). Moreover, the implication is that the nonhomogeneous ordinary differential Eq. (5) also has infinitely many solutions. As a reminder, the aforementioned formulation is based on the assumption that only the temperature profile is known over the entire domain of interest.

Based on the above discussion, a necessary condition for having a unique solution $k(x, t)$ of the ordinary differential Eq. (5) can be stated as follows:

There exists x_0 in the interval [0, 1], such that

$$\frac{\partial T(x_0, \hat{t})}{\partial x} = 0 \tag{9}$$

Furthermore, the unique solution of the nonhomogeneous ordinary differential Eq. (5) also requires the introduction of the following definition:

A function $F(x)$ [e.g., $F(x) = \partial T(x, \hat{t})/\partial x$] is said to have a zero (or root) of order 1 at x_0 if $F(x)$ can be written as

$$F(x) = (x - x_0)G(x) \tag{10}$$

where $G(x)$ is a function and $G(x_0) \neq 0$. This definition implies that if $F(x)$ has a zero of order 1 at x_0, then $F'(x_0) \neq 0$. The application of this property can be found in the next section for a unique solution of the ordinary differential Eq. (5).

As a reminder, the above discussion is based on the assumption that only the temperature profile is known over the entire domain of interest. However, this constraint can be relaxed if both temperature data and surface heat flux at either boundary are known. The constant C in Eq. (8) can be replaced by the surface heat flux. Equation (9) can then be eliminated as the necessary condition to obtain a unique solution.

Continuous Formulation

The feasibility of using a finite difference technique to determine the thermal conductivity from the temperature profile of the governing Eq. (1) is examined in this section. At $t = \hat{t}$, the corresponding governing equation can be expanded as

$$\frac{\partial T(x, \hat{t})}{\partial x} \cdot \frac{\partial k(x, \hat{t})}{\partial x} + \frac{\partial^2 T(x, \hat{t})}{\partial x^2} \cdot k(x, \hat{t}) = \frac{\partial T(x, \hat{t})}{\partial t} - g(x, \hat{t}) \tag{11}$$

Suppose $\partial T(x, \hat{t})/\partial x$ is nonzero over [0, 1], except at x_0 has a zero of order 1 where $0 \leq x_0 \leq 1$, then we have

$$\frac{\partial^2 T(x_0, \hat{t})}{\partial x^2} \cdot k(x_0, \hat{t}) = \frac{\partial T(x_0, \hat{t})}{\partial t} - g(x_0, \hat{t}) \tag{12}$$

Since $\partial^2 T(x_0, \hat{t})/\partial x^2 \neq 0$, one can compute k at $(x_0, \hat{t})$ as

$$k(x_0, \hat{t}) = \left[\frac{\partial T(x_0, \hat{t})}{\partial t} - g(x_0, \hat{t})\right] \Big/ \frac{\partial^2 T(x_0, \hat{t})}{\partial x^2} \tag{13}$$

At this point, one may use the point x_0 to divide the original interval [0, 1] into two subintervals $[0, x_0]$ and $[x_0, 1]$. Applying any stiff ordinary differential equation solver (e.g., backward Euler's method) to the nonhomogeneous ordinary differential Eq. (11) over the subinterval $[x_0, 1]$ with the initial condition specified by Eq. (13), one can approximate $k(x, \hat{t})$ over the subinterval $[x_0, 1]$. Subsequently, we can apply the same stiff ordinary differential equation solver to the nonhomogeneous ordinary differential Eq. (11) over the subinterval $[0, x_0]$ with the same initial condition (i.e., solving the ordinary differential equation backward in x) to obtain an approximation of $k(x, \hat{t})$ over the subinterval $[0, x_0]$.

In order to demonstrate how the approximation works, the backward Euler's method is applied to the differential Eq. (11) over the subinterval $[x_0, 1]$. First, we discretize the interval $[x_0, 1]$ with mesh width Δx and grid points $x_n = x_0 + n \cdot \Delta x$ ($n = 1, 2, \ldots, N$, and $N \cdot \Delta x = 1 - x_0$). Then we use k_0 to denote the initial value of the thermal conductivity evaluated at $(x_0, \hat{t})$, and k_n for the approximated value of k at $(x_n, \hat{t})$, where $n = 1, 2, \ldots, N$ and $N \cdot \Delta x = 1 - x_0$. Once the value of k_{n-1} ($n = 1, 2, \ldots, N$) is available, then we can proceed to compute k_n. The idea of the backward Euler's method is to evaluate the ordinary differential equation at the point $(x_n, \hat{t})$ and use backward differencing to approximate $\partial k/\partial x$ at $(x_n, \hat{t})$. Based on the aforementioned discussion, it yields the following:

$$\frac{\partial T(x_n, \hat{t})}{\partial x} \cdot \frac{k_n - k_{n-1}}{\Delta x} + \frac{\partial^2 T(x_n, \hat{t})}{\partial x^2} \cdot k_n = \frac{\partial T(x_n, \hat{t})}{\partial t} - g(x_n, \hat{t}) \tag{14}$$

Thus

$$k_n = \frac{1}{1 + \Delta x \cdot \dfrac{\dfrac{\partial^2 T(x_n, \hat{t})}{\partial x^2}}{\dfrac{\partial T(x_n, \hat{t})}{\partial x}}} \times \left\{ k_{n-1} + \frac{\Delta x}{\dfrac{\partial T(x_n, \hat{t})}{\partial x}} \left[\frac{\partial T(x_n, \hat{t})}{\partial t} - g(x_n, \hat{t})\right] \right\} \tag{15}$$

For simplicity, suppose we also discretize the interval $[0, x_0]$ with mesh Δx and grid points $x_{-n} = x_0 - n \cdot \Delta x$ (where $n = 1, 2, \ldots, M$, and $M \cdot \Delta x = x_0$). Let k_{-n} denote the approximated value of k at $(x_{-n}, \hat{t})$, one can obtain the following approximation for the ordinary differential Eq. (11) over the subinterval $[0, x_0]$

$$k_{-n} = \frac{1}{-1 + \Delta x \cdot \dfrac{\dfrac{\partial^2 T(x_{-n}, \hat{t})}{\partial x^2}}{\dfrac{\partial T(x_{-n}, \hat{t})}{\partial x}}} \times \left\{ -k_{-n+1} + \frac{\Delta x}{\dfrac{\partial T(x_{-n}, \hat{t})}{\partial x}} \left[\frac{\partial T(x_{-n}, \hat{t})}{\partial t} - g(x_{-n}, \hat{t})\right] \right\} \tag{16}$$

The above discussion is based on the assumption that only the temperature profile is known. However, if we also know the heat flux q at the point $(x_0, \hat{t})$, we can easily compute the initial thermal conductivity k at this location as

$$k(x_0, \hat{t}) = q \Big/ \frac{\partial T(x_0, \hat{t})}{\partial x} \tag{17}$$

instead of using Eq. (13).

In an experimental apparatus to measure thermal conductivity, the boundary heat flux is often generated by dissipation of electrical power in a resistor. By measuring the voltage and current at the surface, the initial thermal conductivity k_0 can be calculated from the Fourier heat conduction equation by expressing the heat flux as the product of the voltage and current.

Computational Algorithm—Continuous Formulation

In summary, the thermal conductivity can be calculated by solving the inverse heat conduction at $t = \hat{t}$, with the assumption that $T(x, t)$ is known over the entire domain. The calculation procedure is comprised of the following steps:

1) If $\partial T(x, \hat{t})/\partial x$ has a zero of order 1 at x_0, where $x_0 \in [0, 1]$, then

$$k_0 = \left[\frac{\partial T(x_0, \hat{t})}{\partial t} - g(x_0, \hat{t})\right] \Big/ \frac{\partial^2 T(x_0, \hat{t})}{\partial x^2} \tag{18a}$$

otherwise

$$k_0 = q \Big/ \frac{\partial T(x_0, \hat{t})}{\partial x} \tag{18b}$$

2) For $n = 1, 2, \ldots, N$, where $N \cdot \Delta x = 1 - x_0$, compute

$$k_n = \frac{1}{1 + \Delta x \cdot \dfrac{\dfrac{\partial^2 T(x_n, \hat{t})}{\partial x^2}}{\dfrac{\partial T(x_n, \hat{t})}{\partial x}}} \times \left\{ k_{n-1} + \frac{\Delta x}{\dfrac{\partial T(x_n, \hat{t})}{\partial x}} \left[\frac{\partial T(x_n, \hat{t})}{\partial t} - g(x_n, \hat{t})\right]\right\} \tag{19}$$

3) For $n = 1, 2, \ldots, M$, where $M \cdot \Delta x = x_0$, compute

$$k_{-n} = \frac{1}{-1 + \Delta x \cdot \dfrac{\dfrac{\partial^2 T(x_{-n}, \hat{t})}{\partial x^2}}{\dfrac{\partial T(x_{-n}, \hat{t})}{\partial x}}} \times \left\{ -k_{-n+1} + \frac{\Delta x}{\dfrac{\partial T(x_{-n}, \hat{t})}{\partial x}} \left[\frac{\partial T(x_{-n}, \hat{t})}{\partial t} - g(x_{-n}, \hat{t})\right]\right\} \tag{20}$$

Then we have the approximated value for the thermal conductivity at $t = \hat{t}$.

Discrete Formulation

In this section, another numerical procedure based on discrete temperatures is presented for the inverse determination of the thermal conductivity. We discretize the entire domain $\{(x, t): x \in [0, 1]$ and $t \in [0, \infty)\}$ with mesh width Δx and Δt, grid points $x_j = j \cdot \Delta x$ (where $j = 0, 1, \ldots, N$ and $N \cdot \Delta x = 1$) and $t_i = i \cdot \Delta t$ ($i = 0, 1, 2, \ldots$). The present procedure will drop the assumption that the temperature $T(x, t)$ is known over the entire domain as mentioned in the previous section. Instead, it only assumes the temperature $T(x, t)$ is known at those grid points. Suppose one is interested in recovering the conductivity $k(x, t)$ at $t = t_{i_0}$, the basic requirement is that the temperature measurements of $T(x, t)$ at $t = t_{i_0}$ and $t = t_{i_0+1}$ must be available. Once this condition has been satisfied, one can approximate $\partial T(x_j, t_{i_0})/\partial t$ ($j = 0, 1, \ldots, N$) by forward differencing. In addition, we can use central differencing to approximate $\partial T(x_j, t_{i_0})/\partial x$ and $\partial^2 T(x_j, t_{i_0})/\partial x^2$ for $j = 1, 2, \ldots, N - 1$. At the boundary, i.e., $j = 0$ and $j = N$, one can use forward and backward differencing, respectively, to approximate $\partial T(x_j, t_{i_0})/\partial x$ and $\partial^2 T(x_j, t_{i_0})/\partial x^2$. As a result of these calculations, all the approximated coefficients of the nonhomogeneous ordinary differential Eq. (11) are available at the grid points. Furthermore, suppose there exists an integer M where $0 < M < N$, such that the approximated $\partial T(x_M, t_{i_0})/\partial x = 0$ and the approximated $\partial^2 T(x_M, t_{i_0})/\partial x^2 \neq 0$. (The exact criterion for determining x_M can be found in the Computational Algorithm—Discrete Formulation section). Now, one can approximate k_M by using Eq. (18a) with $\hat{t}$ replaced by t_{i_0}, and the subscript 0 replaced by M. Also, $\partial T/\partial t$ and $\partial^2 T/\partial^2 x$ at (x_M, t_{i_0}) are the approximated values.

By knowing k_M, a path similar to the continuous solution is followed and the original problem is broken into two ordinary differential equations. Denote k_j as the approximated value of k at (x_j, t_{i_0}), we can then use Eq. (19) to recover k_j for $j = M + 1, \ldots, N$ and Eq. (20) to recover k_j for $j = M - 1, \ldots, 0$.

Similar to the continuous case, the above discussion is based on the assumption that only the temperature profile is known. However, if we also know the heat flux q at the point (x_M, t_{i_0}), we can easily compute the initial thermal conductivity k at this particular location. (In practice, the heat flux q can only be measured at the boundary, e.g., $x_M = 0$ or $x_M = 1$).

Computational Algorithm—Discrete Formulation

The numerical procedure for the determination of the thermal conductivity is summarized below. We are interested in solving the inverse heat conduction at $t = t_{i_0}$ with the assumption that $T(x, t)$ is known only at the grid points. The algorithm consists of the following steps:

1) For $n = N$, compute

$$\frac{\overline{\partial T}(x_N, t_{i_0})}{\partial t} = \frac{T(x_N, t_{i_0+1}) - T(x_N, t_{i_0})}{\Delta t} \tag{21}$$

$$\frac{\overline{\partial T}(x_N, t_{i_0})}{\partial x} = \frac{T(x_N, t_{i_0}) - T(x_{N-1}, t_{i_0})}{\Delta x} \tag{22}$$

$$\frac{\overline{\partial^2 T}(x_N, t_{i_0})}{\partial x^2} = \frac{T(x_N, t_{i_0}) - 2T(x_{N-1}, t_{i_0}) + T(x_{N-2}, t_{i_0})}{\Delta x^2} \tag{23}$$

2) For $n = 0$, compute

$$\frac{\overline{\partial T}(x_0, t_{i_0})}{\partial t} = \frac{T(x_0, t_{i_0+1}) - T(x_0, t_{i_0})}{\Delta t} \tag{24}$$

$$\frac{\overline{\partial T}(x_0, t_{i_0})}{\partial x} = \frac{T(x_1, t_{i_0}) - T(x_0, t_{i_0})}{\Delta x} \tag{25}$$

$$\frac{\overline{\partial^2 T}(x_0, t_{i_0})}{\partial x^2} = \frac{T(x_2, t_{i_0}) - 2T(x_1, t_{i_0}) + T(x_0, t_{i_0})}{\Delta x^2} \tag{26}$$

3) For $n = 1, \ldots, N - 1$, compute

$$\frac{\overline{\partial T}(x_n, t_{i_0})}{\partial t} = \frac{T(x_n, t_{i_0+1}) - T(x_n, t_{i_0})}{\Delta t} \tag{27}$$

$$\frac{\overline{\partial T}(x_n, t_{i_0})}{\partial x} = \frac{T(x_{n+1}, t_{i_0}) - T(x_{n-1}, t_{i_0})}{2\Delta x} \tag{28}$$

$$\frac{\overline{\partial^2 T}(x_n, t_{i_0})}{\partial x^2} = \frac{T(x_{n+1}, t_{i_0}) - 2T(x_n, t_{i_0}) + T(x_{n-1}, t_{i_0})}{\Delta x^2} \tag{29}$$

4) For $n = 1, \ldots, N - 1$: If $\overline{\partial T}(x_n, t_{i_0})/\partial x = 0$ and $\overline{\partial^2 T}(x_n, t_{i_0})/\partial x^2 \neq 0$, i.e., $T(x_{n+1}, t_{i_0}) - T(x_{n-1}, t_{i_0}) = 0$ and $T(x_{n+1}, t_{i_0}) - 2T(x_n, t_{i_0}) + T(x_{n-1}, t_{i_0}) \neq 0$. Then $x_M = x_n$ and

$$k_M = \left[\frac{\overline{\partial T}(x_M, t_{i_0})}{\partial t} - g(x_M, t_{i_0})\right] \Big/ \frac{\overline{\partial^2 T}(x_M, t_{i_0})}{\partial x^2} \tag{30a}$$

If such n does not exist, then

$$k_M = q \Big/ \frac{\overline{\partial T}(x_M, t_{i_0})}{\partial x} \tag{30b}$$

where q is the known heat flux at (x_M, t_{i_0}).

5) For $n = M + 1, \ldots, N$, compute

$$k_n = \frac{1}{1 + \Delta x \cdot \dfrac{\dfrac{\partial^2 T(x_n, t_{i_0})}{\partial x^2}}{\dfrac{\partial T(x_n, t_{i_0})}{\partial x}}} \times \left\{ k_{n-1} + \frac{\Delta x}{\dfrac{\partial T(x_n, t_{i_0})}{\partial x}} \left[\frac{\partial T(x_n, t_{i_0})}{\partial t} - g(x_n, t_{i_0}) \right] \right\} \tag{31}$$

6) For $n = M - 1, \ldots, 0$, compute

$$k_n = \frac{1}{-1 + \Delta x \cdot \dfrac{\dfrac{\partial^2 T(x_n, t_{i_0})}{\partial x^2}}{\dfrac{\partial T(x_n, t_{i_0})}{\partial x}}} \times \left\{ -k_{n+1} + \frac{\Delta x}{\dfrac{\partial T(x_n, t_{i_0})}{\partial x}} \left[\frac{\partial T(x_n, t_{i_0})}{\partial t} - g(x_n, t_{i_0}) \right] \right\} \tag{32}$$

Then we have an approximated value for the thermal conductivity at $t = t_{i0}$.

In the discrete case, the exact value of $\partial T/\partial x$ is not available. Therefore, we use finite differencing to approximate the temperature gradients $\partial T/\partial x$ and $\partial^2 T/\partial x^2$ (refer to item 4 for details). It can easily be shown that

$$\left| \frac{\partial T(x_M, t_{i_0})}{\partial x} - \frac{\partial T(\alpha, t_{i_0})}{\partial x} \right| = \mathcal{O}(\Delta x^2) \tag{33}$$

where α is the root of $\partial T(x, t_{i_0})/\partial x$. We conclude that the error associated with x_M caused by interpolating the temperature gradients is $\mathcal{O}(\Delta x^2)$.

Stability and Accuracy of the Numerical Procedures

Understanding and controlling the numerical error is essential for a successful solution of the finite difference equation. In this section, the choice of using the backward Euler's method is first addressed. The stability and accuracy requirements that lead to the convergence of the numerical procedures are then discussed.

Method of Solution

For both the continuous and discrete formulations, we assume $\partial T/\partial x$ has a zero of order 1 at $(x_0, \hat{t})$ or (x_M, t_{i_0}), respectively. Let us concentrate on the continuous case in the following discussion. It has been proven previously that one can solve for the conductivity at the point $(x_0, \hat{t})$ based on this information. Additionally, due to the selection of x_0 and the breakup of the interval [0, 1], we have two initial value problems over the intervals $[0, x_0]$ and $[x_0, 1]$. Currently, there are many ordinary differential equation solvers[24] available to solve such problems. The backward Euler's method is chosen in this study for the following reasons:

1) Easy to implement: the backward Euler's method is one of the simplest methods available for stiff problems.

2) A-Stable numerical scheme: a desirable property for the convergence of the numerical procedure.

3) Practicality: in reality, one may not be able to use too many temperature sensors during the experiment. As a result, Δx cannot be too small. Usually, ordinary differential equation solvers are very sensitive to the mesh size Δx. However, using the backward Euler's method will reduce the sensitivity.

Stability Analysis—Continuous Case

First, let us address the order of accuracy of the numerical procedure for the continuous case. Since we use the exact value of $\partial T/\partial t$ and $\partial^2 T/\partial^2 x$ in Eq. (18) to compute k_0, the initial value in this case does not generate any error. In fact, the error for the numerical procedure only comes from the backward Euler's method. Let $k(x_n, \hat{t})$ be the true conductivity, then the local truncation error (LTE) for the conductivity is

$$\begin{aligned}
\text{LTE} &= \frac{\partial T(x_n, \hat{t})}{\partial x} [k(x_n, \hat{t}) - k(x_{n-1}, \hat{t})] \\
&\quad + \frac{\partial^2 T(x_n, \hat{t})}{\partial x^2} \cdot k(x_n, \hat{t}) \cdot \Delta x - \frac{\partial T(x_n, \hat{t})}{\partial t} \cdot \Delta x \\
&\quad + g(x_n, \hat{t}) \cdot \Delta x \\
&= \frac{\partial T(x_n, \hat{t})}{\partial x} \left[\frac{\partial k(x_n, \hat{t})}{\partial x} \cdot \Delta x + \mathcal{O}(\Delta x^2) \right] \\
&\quad + \frac{\partial^2 T(x_n, \hat{t})}{\partial x^2} \cdot k(x_n, \hat{t}) \cdot \Delta x - \frac{\partial T(x_n, \hat{t})}{\partial t} \cdot \Delta x \\
&\quad + g(x_n, \hat{t}) \cdot \Delta x \\
&= \frac{\partial T(x_n, \hat{t})}{\partial x} \cdot \frac{\partial k(x_n, \hat{t})}{\partial x} \cdot \Delta x + \mathcal{O}(\Delta x^2) \\
&\quad + \frac{\partial^2 T(x_n, \hat{t})}{\partial x^2} \cdot k(x_n, \hat{t}) \cdot \Delta x - \frac{\partial T(x_n, \hat{t})}{\partial t} \cdot \Delta x \\
&\quad + g(x_n, \hat{t}) \cdot \Delta x \\
&= \mathcal{O}(\Delta x^2)
\end{aligned} \tag{34}$$

This shows the order of accuracy of the numerical procedure for the continuous case is 1 and the local truncation error is $\mathcal{O}(\Delta x^2)$. It is well-known that the backward Euler's method is A-stable and consistent (Ref. 24, Chap. 11); implying convergence for the continuous case.

Stability Analysis—Discrete Case

For this case, the discussion will focus on the interval $[x_M, 1]$. A similar argument for the order of accuracy on the interval $[0, x_M]$ is also applied. By using the forward difference and central difference schemes to approximate the temperature derivatives $\partial T/\partial t$ and $\partial^2 T/\partial^2 x$, these two terms take the forms:

$$\frac{\partial T(x_N, t_{i_0})}{\partial t} = \frac{T(x_M, t_{i_0+1}) - T(x_M, t_{i_0})}{\Delta t} + \mathcal{O}(\Delta t) \tag{35}$$

$$\frac{\partial^2 T(x_M, t_{i_0})}{\partial x^2} = \frac{T(x_{M+1}, t_{i_0}) - 2T(x_M, t_{i_0}) + T(x_{M-1}, t_{i_0})}{\Delta x^2} + \mathcal{O}(\Delta x^2) \tag{36}$$

By substituting into Eq. (30a), one can obtain

$$\begin{aligned}
k_M &= \frac{\dfrac{\partial T(x_M, t_{i_0})}{\partial t} + \mathcal{O}(\Delta t) - g(x_M, t_{i_0})}{\dfrac{\partial^2 T(x_M, t_{i_0})}{\partial x^2} + \mathcal{O}(\Delta x^2)} \\
&= \frac{\dfrac{\partial T(x_M, t_{i_0})}{\partial t} - g(x_M, t_{i_0})}{\dfrac{\partial^2 T(x_M, t_{i_0})}{\partial x^2}} + \mathcal{O}(\Delta t) + \mathcal{O}(\delta x^2)
\end{aligned} \tag{37}$$

This implies $k(x_M, t_{i_0}) - k_M = \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2)$. If we let $\Delta t = \Delta x^2$, one can show that the truncation error for the initial value is $\mathcal{O}(\Delta x^2)$.

For $M < n < N$, we use forward difference to approximate $\partial T/\partial t$, and central difference to approximate $\partial T/\partial x$ and $\partial^2 T/\partial x^2$. These temperature derivatives can be expressed by the following finite difference equations:

$$\frac{\partial T(x_n, t_{i_0})}{\partial t} = \frac{T(x_n, t_{i_0+1}) - T(x_n, t_{i_0})}{\Delta t} + \mathcal{O}(\Delta t) \quad (38)$$

$$\frac{\partial T(x_n, t_{i_0})}{\partial x} = \frac{T(x_{n+1}, t_{i_0}) - T(x_{n-1}, t_{i_0})}{2\Delta x} + \mathcal{O}(\Delta x^2) \quad (39)$$

$$\frac{\partial^2 T(x_n, t_{i_0})}{\partial x^2} = \frac{T(x_{n+1}, t_{i_0}) - 2T(x_n, t_{i_0}) + T(x_{n-1}, t_{i_0})}{\Delta x^2} + \mathcal{O}(\Delta x^2) \quad (40)$$

Let $k(x_n, t_{i_0})$ be the true conductivity at the location (x_n, t_{i_0}). Then the LTE for the conductivity is

$$\begin{aligned}
\text{LTE} &= \left[\frac{\partial T(x_n, t_{i_0})}{\partial x} + \mathcal{O}(\Delta x^2)\right][k(x_n, t_{i_0}) - k(x_{n-1}, t_{i_0})] \\
&+ \left[\frac{\partial^2 T(x_n, t_{i_0})}{\partial x^2} + \mathcal{O}(\Delta x^2)\right]\cdot k(x_n, t_{i_0})\cdot \Delta x \\
&- \left[\frac{\partial T(x_n, t_{i_0})}{\partial t} + \mathcal{O}(\Delta t)\right]\cdot \Delta x + g(x_n, t_{i_0})\cdot \Delta x \\
&= \left[\frac{\partial T(x_n, t_{i_0})}{\partial x} + \mathcal{O}(\Delta x^2)\right]\left[\Delta x\cdot \frac{\partial k(x_n, t_{i_0})}{\partial x} + \mathcal{O}(\Delta x^2)\right] \\
&+ \Delta x\cdot \frac{\partial^2 T(x_n, t_{i_0})}{\partial x^2}\cdot k(x_n, t_{i_0}) + \mathcal{O}(\Delta x^3) \\
&- \Delta x\cdot \frac{\partial T(x_n, t_{i_0})}{\partial x} + \mathcal{O}(\Delta t\cdot \Delta x) + g(x_n, t_{i_0})\cdot \Delta x \\
&= \Delta x\cdot \frac{\partial T(x_n, t_{i_0})}{\partial x}\cdot \frac{\partial k(x_n, t_{i_0})}{\partial x} + \mathcal{O}(\Delta x^2) \\
&+ \Delta x\cdot \frac{\partial^2 T(x_n, t_{i_0})}{\partial x^2}\cdot k(x_n, t_{i_0}) + \mathcal{O}(\Delta x^3) \\
&- \Delta x\cdot \frac{\partial T(x_n, t_{i_0})}{\partial t} + \mathcal{O}(\Delta t\cdot \Delta x) + g(x_n, t_{i_0})\cdot \Delta x \\
&= \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t\cdot \Delta x)
\end{aligned} \quad (41)$$

If we let $\Delta t = \Delta x^2$ in Eq. (41), then the LTE $= \mathcal{O}(\Delta x^2)$. Note that in the process of the derivation above, the LTE has been multiplied by the mesh size Δx. This implies the order of accuracy of the numerical procedure for the interior points x_n $(M < n < N)$ is first order.

For $n = N$, we use forward difference to approximate $\partial T/\partial t$, and backward difference to approximate $\partial T/\partial x$ and $\partial^2 T/\partial x^2$. As a result, it yields the following finite difference equations for the temperature derivatives:

$$\frac{\partial T(x_N, t_{i_0})}{\partial t} = \frac{T(x_N, t_{i_0+1}) - T(x_N, t_{i_0})}{\Delta t} + \mathcal{O}(\Delta t) \quad (42)$$

$$\frac{\partial T(x_N, t_{i_0})}{\partial x} = \frac{T(x_N, t_{i_0}) - T(x_{N-1}, t_{i_0})}{\Delta x} + \mathcal{O}(\Delta x) \quad (43)$$

$$\frac{\partial^2 T(x_N, t_{i_0})}{\partial x^2} = \frac{T(x_N, t_{i_0}) - 2T(x_{N-1}, t_{i_0}) + T(x_{N-2}, t_{i_0})}{\Delta x^2} + \mathcal{O}(\Delta x) \quad (44)$$

Let $k(x_N, t_{i_0})$ be the true conductivity at the location (x_N, t_{i_0}). Then the LTE is

$$\begin{aligned}
\text{LTE} &= \left[\frac{\partial T(x_N, t_{i_0})}{\partial x} + \mathcal{O}(\Delta x)\right][k(x_N, t_{i_0}) - k(x_{N-1}, t_{i_0})] \\
&+ \left[\frac{\partial^2 T(x_N, t_{i_0})}{\partial x^2} + \mathcal{O}(\Delta x)\right]\cdot k(x_N, t_{i_0})\cdot \Delta x \\
&- \left[\frac{\partial T(x_N, t_{i_0})}{\partial t} + \mathcal{O}(\Delta t)\right]\cdot \Delta x + g(x_N, t_{i_0})\cdot \Delta x \\
&= \left[\frac{\partial T(x_N, t_{i_0})}{\partial x} + \mathcal{O}(\Delta x)\right]\left[\Delta x\cdot \frac{\partial k(x_N, t_{i_0})}{\partial x} + \mathcal{O}(\Delta x^2)\right] \\
&+ \Delta x\cdot \frac{\partial^2 T(x_N, t_{i_0})}{\partial x^2}\cdot k(x_N, t_{i_0}) + \mathcal{O}(\Delta x^2) \\
&- \Delta x\cdot \frac{\partial T(x_N, t_{i_0})}{\partial t} + \mathcal{O}(\Delta t\cdot \Delta x) + g(x_N, t_{i_0})\cdot \Delta x \\
&= \Delta x\cdot \frac{\partial T(x_N, t_{i_0})}{\partial x}\cdot \frac{\partial k(x_N, t_{i_0})}{\partial x} + \mathcal{O}(\Delta x^2) \\
&+ \Delta x\cdot \frac{\partial^2 T(x_N, t_{i_0})}{\partial x^2}\cdot k(x_N, t_{i_0}) + \mathcal{O}(\Delta x^2) \\
&- \Delta x\cdot \frac{\partial T(x_N, t_{i_0})}{\partial t} + \mathcal{O}(\Delta t\cdot \Delta x) + g(x_N, t_{i_0})\cdot \Delta x \\
&= \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t\cdot \Delta x)
\end{aligned} \quad (45)$$

This implies the order of accuracy for the end point is also first-order since the LTE has been multiplied by the mesh size Δx in the derivation. Hence, the numerical procedure has first-order accuracy in the interval $[x_0, 1]$. A similar statement is true for the interval $[x_0, 1]$. Since the ordinary differential equation is linear, the discrete case is convergent.

Numerical Experiments

To show the applicability of the proposed finite difference approximation procedures for the inverse determination of the thermal conductivity in a one-dimensional heat conduction domain, three distinct example problems were solved. Examples include constant, spatially dependent, or temperature dependent quantities that are reconstructed from available continuous or discrete temperature measurement data. The algorithms were programmed in Fortran 77, and the numerical results were obtained in double-precision arithmetic. Computations were performed using a SUN Sparc10 Workstation.

The exact temperature and thermal conductivity used in the following examples are preselected profiles such that these functions satisfy the governing heat conduction equation and the boundary conditions as well as the initial condition. The simulated temperature data are generated from the preselected temperature profiles. The numerical procedure will be tested by computing the thermal conductivity from these preselected temperatures. The accuracy of the procedure will be assessed by comparing the calculated results with the preselected thermal conductivity profiles.

Continuous Case

The following three examples illustrate the application of the proposed numerical procedure based on the computational algorithm given by Eqs. (18–20) for the continuous formulation. The calculations are performed assuming the temperature profiles are available over the entire domain, $0 \le x \le 1$.

Example C1—Constant Thermal Conductivity

A slab, $0 \le x \le 1$, is initially at temperature $T(x) = \sin(\pi x)$. For time $t > 0$, the boundaries (at $x = 0$ and $x = 1$) are kept at zero temperatures. The mathematical formulation of this problem is given as

$$\frac{\partial T(x, t)}{\partial t} - \frac{\partial}{\partial x}\left[k(x, t)\frac{\partial T(x, t)}{\partial x}\right] = g(x, t)$$

$$\text{in} \quad 0 < x < 1, \qquad t > 0 \tag{46a}$$

$$\text{B.C.} \quad T|_{x=0} = 0, \qquad t > 0 \tag{46b}$$

$$T|_{x=1} = 0, \qquad t > 0 \tag{46c}$$

$$\text{I.C.} \quad T|_{t=0} = \sin(\pi x), \qquad 0 \le x \le 1 \tag{46d}$$

Heat generation

$$g(x, t) = 0 \tag{46e}$$

The exact solutions for the temperature distribution and thermal conductivity in the slab are

$$T(x, t) = e^{-2\pi^2 t}\sin(\pi x) \tag{46f}$$

$$k(x, t) = 2 \tag{46g}$$

To determine the thermal conductivity the spatial interval $0 \le x \le 1$ is divided into $N = 10$ intervals. The iteration step corresponds to a mesh size of $\Delta x = 0.1$. The initial value for the thermal conductivity was evaluated at the location $x = 0.5$, where the condition $\partial T/\partial x = 0$ is satisfied. The numerical results of the thermal conductivities are tabulated in Table 1 for time $t = 0.2$. Clearly, the results from the present study are in excellent agreement with the exact solutions. The solutions obtained by using $\Delta x = 0.1$ are exactly the same as the exact solution.

Example C2—Spatially-Dependent Thermal Conductivity

A slab, $0 \le x \le 1$, is initially at zero temperature. For time $t > 0$, heat is generated in the solid at a variable rate of $g(x, t)$, the boundaries at $x = 0$ and $x = 1$ are subjected to time-varied temperatures $T(t) = 0.36te^{-t}$ and $T(t) = 0.16te^{-t}$, respectively. The mathematical formulation of this problem is given as

$$\frac{\partial T(x, t)}{\partial t} - \frac{\partial}{\partial x}\left[k(x, t)\frac{\partial T(x, t)}{\partial x}\right] = g(x, t)$$

$$\text{in} \quad 0 < x < 1, \qquad t > 0 \tag{47a}$$

$$\text{B.C.} \quad T|_{x=0} = 0.36te^{-t}, \qquad t > 0 \tag{47b}$$

$$T|_{x=1} = 0.16te^{-t}, \qquad t > 0 \tag{47c}$$

$$\text{I.C.} \quad T|_{t=0} = 0, \qquad 0 \le x \le 1 \tag{47d}$$

Heat generation

$$g(x, t) = (x - 0.6)^2(1 - t)e^{-t} - \{2 + [0.5 - 4(x - 0.3)(x - 0.6)]e^{-4(x-0.3)^2}\}te^{-t} \tag{47e}$$

The exact solutions for the temperature distribution and thermal conductivity in the slab are

$$T(x, t) = (x - 0.6)^2 te^{-t} \tag{47f}$$

$$k(x, t) = 1 + 0.25e^{-4(x-0.3)^2} \tag{47g}$$

To determine the thermal conductivity the spatial coordinate $0 \le x \le 1$ is divided into $N = 10, 20, 40$ intervals that correspond to $\Delta x = 0.100$, 0.050, and 0.025, respectively. The initial value for the thermal conductivity was evaluated at $x = 0.6$ where the condition $\partial T/\partial x = 0$ is satisfied. The results are tabulated in Table 2 for time $t = 0.2$. The calculated thermal conductivities from the present study are compared with the exact solutions. Clearly, these numerical results are in close agreement with the existing work. By decreasing the grid size used in the calculation, the accuracy of the approximation increases. After examining the maximum errors for various mesh sizes, one can find that the convergent rate of the proposed numerical procedure is first order.

Table 1 Estimated thermal conductivities for example C1 with iteration step $\Delta x = 0.1$

	k	
x	Exact solution	Present study
0.0	2.0000	2.0000
0.1	2.0000	2.0000
0.2	2.0000	2.0000
0.3	2.0000	2.0000
0.4	2.0000	2.0000
0.5	2.0000	2.0000
0.6	2.0000	2.0000
0.7	2.0000	2.0000
0.8	2.0000	2.0000
0.9	2.0000	2.0000
1.0	2.0000	2.0000
Maximum error	——	0.0000

Table 2 Estimated thermal conductivities for example C2 with iterations N = 10, 20, and 40 at t = 0.2

	k			
		Present study		
x	Exact solution	Δx = 0.100	Δx = 0.050	Δx = 0.025
0.0	1.1744	1.1542	1.1641	1.1692
0.1	1.2130	1.1927	1.2027	1.2079
0.2	1.2402	1.2227	1.2314	1.2358
0.3	1.2500	1.2375	1.2437	1.2469
0.4	1.2402	1.2334	1.2368	1.2385
0.5	1.2130	1.2108	1.2119	1.2125
0.6	1.1744	1.1744	1.1744	1.1744
0.7	1.1318	1.1320	1.1319	1.1319
0.8	1.0920	1.0942	1.0931	1.0925
0.9	1.0592	1.0641	1.0617	1.0605
1.0	1.0352	1.0425	1.0389	1.0371
Maximum error	——	0.0204	0.0105	0.0053

Example C3—Temperature-Dependent Thermal Conductivity

A slab, $0 \le x \le 1$, is initially at a temperature $T(x) = \cos \pi(x - 0.8)$. For time $t > 0$, heat is generated in the solid at a variable rate of $g(x, t)$. The normal derivative of temperature is prescribed at the boundary $x = 0$, while the boundary $x = 1$ dissipates heat by convection into an environment of zero temperature. Both conditions vary with time along the surfaces. The mathematical formulation of this problem is given as

$$\frac{\partial T(x,t)}{\partial t} - \frac{\partial}{\partial x}\left[k(x,t)\frac{\partial T(x,t)}{\partial x}\right] = g(x,t)$$
$$\text{in} \quad 0 < x < 1, \qquad t > 0 \tag{48a}$$

$$\text{B.C.} \left.\frac{\partial T}{\partial x}\right|_{x=0} = \pi e^{-\pi^2 t} \sin 0.8\pi, \qquad t > 0 \tag{48b}$$

$$\left.\left[\frac{\partial T}{\partial x} + T\right]\right|_{x=1} = [\cos 0.2\pi - \pi \sin 0.2\pi]e^{-\pi^2 t}, \qquad t > 0 \tag{48c}$$

$$\text{I.C.} \; T|_{t=0} = \cos \pi(x - 0.8), \qquad 0 \le x \le 1 \tag{48d}$$

Heat generation

$$g(x,t) = -\pi^2 e^{-\pi^2 t} \cos \pi(x - 0.8) + \frac{\pi^2 e^{-\pi^2 t} \cos \pi(x - 0.8)}{1 - e^{-\pi^2 t} \cos \pi(x - 0.8)} - \left[\frac{\pi e^{-\pi^2 t} \sin \pi(x - 0.8)}{1 - e^{-\pi^2 t} \cos \pi(x - 0.8)}\right]^2 \tag{48e}$$

The exact solutions for the temperature distribution and thermal conductivity in the slab are

$$T(x,t) = e^{-\pi^2 t} \cos \pi(x - 0.8) \tag{48f}$$

$$k(x,t) = \frac{1}{1 - T(x,t)} \tag{48g}$$

Again, the spatial coordinate $0 \le x \le 1$ is divided into $N = 10, 20, 40$ intervals in the calculations with a spacing of $\Delta x = 0.100, 0.050$, and 0.025, respectively. The initial value for the thermal conductivity was evaluated at $x = 0.8$, where the condition $\partial T/\partial x = 0$ is satisfied. The results are tabulated in Table 3 for time $t = 0.2$. The inverse solutions for thermal conductivities at various times are shown in Fig. 2. The agreement between the calculated and exact values is very good.

Discrete Case

In this section three examples are used to illustrate the application of the numerical procedure based on the computational algorithm given by Eqs. (21–32) for the discrete formulation. For convenience the example problems from the continuous case are selected. Once the temperature data are known at the grid points, the temperature derivatives used in the discrete formulation are computed at these locations. Finite differences are used to approximate the temperature gradients in the time and space coordinates. The significance of this approach replaces the continuous temperature distribution with a discrete distribution. Hence, the differences between the discrete and continuous cases lie in the calculations of the temperature gradients.

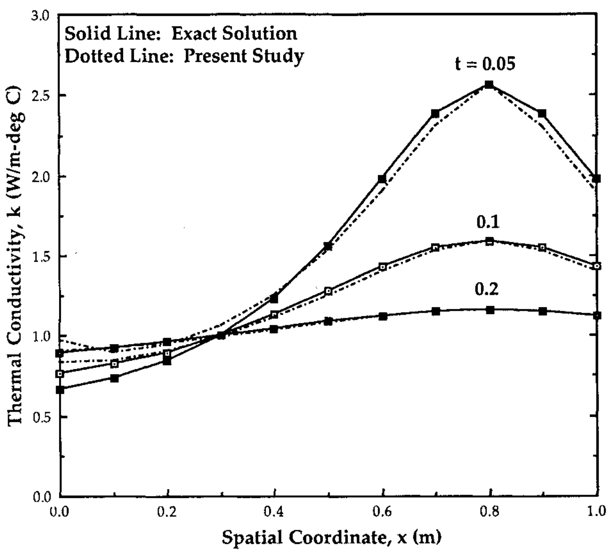

Fig. 2 Comparison of the thermal conductivities for various time intervals (example 3C, $\Delta x = 0.1$).

Table 3 Estimated thermal conductivities for example C3 with iterations $N = 10, 20$, and 40 at $t = 0.2$

	k			
		Present study		
x	Exact solution	$\Delta x =$ 0.100	$\Delta x =$ 0.050	$\Delta x =$ 0.025
0.0	0.8990	0.9019	0.9000	0.8994
0.1	0.9245	0.9214	0.9230	0.9238
0.2	0.9588	0.9523	0.9557	0.9573
0.3	1.0000	0.9911	0.9957	0.9978
0.4	1.0449	1.0347	1.0399	1.0424
0.5	1.0889	1.0790	1.0840	1.0865
0.6	1.1266	1.1186	1.1227	1.1246
0.7	1.1522	1.1477	1.1500	1.1511
0.8	1.1613	1.1613	1.1613	1.1613
0.9	1.1522	1.1477	1.1500	1.1511
1.0	1.1266	1.1186	1.1227	1.1246
Maximum error	——	0.0101	0.0050	0.0025

Table 4 Estimated thermal conductivities for example D1 with iteration step $\Delta x = 0.1$

	k			
		Present study		
x	Exact solution	$\Delta x =$ 0.100	$\Delta x =$ 0.050	$\Delta x =$ 0.025
0.0	2.0000	1.6668	1.9085	1.9765
0.1	2.0000	1.8290	1.9555	1.9887
0.2	2.0000	1.8300	1.9555	1.9887
0.3	2.0000	1.8300	1.9555	1.9887
0.4	2.0000	1.8300	1.9555	1.9887
0.5	2.0000	1.8300	1.9555	1.9887
0.6	2.0000	1.8300	1.9555	1.9887
0.7	2.0000	1.8300	1.9555	1.9887
0.8	2.0000	1.8300	1.9555	1.9887
0.9	2.0000	1.8300	1.9555	1.9887
1.0	2.0000	1.6668	1.9085	1.9765
Maximum error	——	0.3332	0.0915	0.0235

Example D1—Constant Thermal Conductivity

The physical problem is the same as example C1. We used simulated temperature measurement data to calculate the unknown thermal conductivity k. The temperature data are calculated from Eq. (46f) for three mesh sizes, $N = 10, 20$, and 40 intervals. The results are tabulated in Table 4 for time $t = 0.2$.

A comparison of the solution with the continuous case reveals that the error is greater in the discrete case. The error is a consequence of estimations of the temperature derivatives. The maximum error also indicated that the discrete formulation is of order Δx. It is obvious that as the mesh size decreases, the error is reduced and therefore the accuracy of the approximation is increased.

Example D2—Spatially-Dependent Thermal Conductivity

This problem is the same as example C2. Here, the discrete formulation technique is used to solve for the thermal con-

Table 5 Estimated thermal conductivities for example D2 with iterations N = 10, 20, and 40 at t = 0.2

	k			
		Present study		
x	Exact solution	Δx = 0.100	Δx = 0.050	Δx = 0.025
0.0	1.1744	1.1483	1.1626	1.1689
0.1	1.2130	1.1906	1.2022	1.2077
0.2	1.2402	1.2214	1.2311	1.2357
0.3	1.2500	1.2368	1.2436	1.2468
0.4	1.2402	1.2330	1.2367	1.2385
0.5	1.2130	1.2107	1.2119	1.2125
0.6	1.1744	1.1744	1.1744	1.1744
0.7	1.1318	1.1319	1.1319	1.1319
0.8	1.0920	1.0938	1.0930	1.0925
0.9	1.0592	1.0633	1.0615	1.0604
1.0	1.0352	1.0387	1.0380	1.0369
Maximum error	——	0.0262	0.0118	0.0056

Table 6 Estimated thermal conductivities for example D3 with iterations N = 10, 20, and 40 at t = 0.2

	k			
		Present study		
x	Exact solution	Δx = 0.100	Δx = 0.050	Δx = 0.025
0.0	0.8990	0.7486	0.8636	0.8908
0.1	0.9245	0.8784	0.9121	0.9210
0.2	0.9588	0.9104	0.9450	0.9546
0.3	1.0000	0.9500	0.9852	0.9952
0.4	1.0449	0.9944	1.0296	1.0398
0.5	1.0889	1.0393	1.0739	1.0839
0.6	1.1266	1.0795	1.1127	1.1221
0.7	1.1522	1.1090	1.1401	1.1486
0.8	1.1613	1.1227	1.1514	1.1588
0.9	1.1522	1.1090	1.1401	1.1486
1.0	1.1266	1.0107	1.0896	1.1155
Maximum error	——	0.1503	0.0371	0.0111

ductivity. The temperature data at each grid point is obtained by utilizing the exact temperature distribution, Eq. (47f). To determine the thermal conductivity the spatial coordinate $0 \leq x \leq 1$ is divided into N = 10, 20, 40 intervals. The results are tabulated in Table 5 for time t = 0.2. The results compared very well with the exact solutions. For the same reason given in example D1, the error is larger than the continuous case for the same mesh size.

Example D3—Temperature-Dependent Thermal Conductivity

In this case, we consider the same problem corresponding to example C3. The temperature measurement data are calculated from Eq. (48f) for N = 10, 20, 40 intervals. The results are tabulated in Table 6 for time t = 0.2. The inverse solutions for thermal conductivities at various times are shown in Fig. 3. Here, the agreement between the calculated and exact values is very good.

Sensitivity Analysis

The simulated temperature data used in the inverse analysis of thermal conductivity were obtained from the preselected temperature profiles. The measured temperatures would decrease the accuracy of the inverse solutions. In reality, the exact temperature inputs (T_{exact}) used in the above test cases should be modified by adding random errors to simulate experimental measurements (T_{exp})[25]:

$$T_{exp} = T_{exact} - \varepsilon\sigma \tag{49}$$

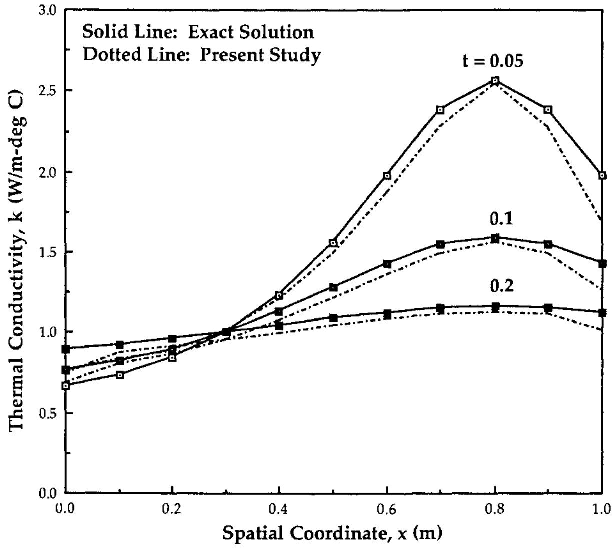

Fig. 3 Comparison of the thermal conductivities for various time intervals (example 3D, Δx = 0.1).

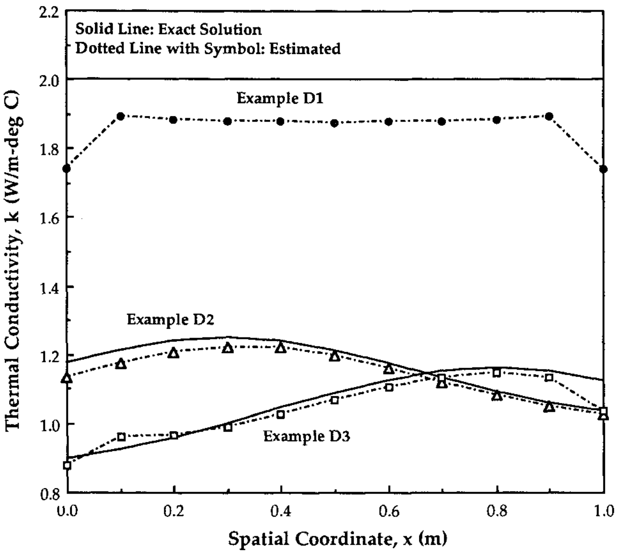

Fig. 4 Effect of temperature measurement error on the determination of thermal conductivity (Δx = 0.1 and t = 0.2).

where σ is the standard deviation of the measurement error that is assumed to be the same for all measurements. For normally distributed errors with zero mean and a 99% confidence, the value of ε lies in the range

$$-2.576 < \varepsilon < 2.576 \tag{50}$$

The product of $\varepsilon\sigma$ represents the temperature measurement errors. In order to test the influence of the experimental errors on the inverse analysis, the test cases previously described will be repeated by incorporating the measurement errors in the simulated temperature measurements. Under the most strict conditions, the simulated temperature data (T_{exp}) are generated by using Eqs. (46f), (47f), and (48f), as well as Eq. (49) with $\sigma = 0.01(T_{exact})_{max}$ and ε = 2.576 or −2.576 in the inverse analysis. The computations are performed based on the discrete formulation.

The effects of the inexact measurements on the inverse analysis are shown in Fig. 4. Clearly, the estimated thermal conductivities are in good agreement with the exact profiles.

Conclusions

Two direct finite difference procedures have been introduced for the inverse determination of the thermal conductivity in a one-dimensional heat conduction domain. The unknown thermal conductivity is reconstructed using available temperature measurements, in the form of either continuous or discrete data. Several heat conduction problems have been tested with the techniques. The estimated thermal conductivities were verified by comparing the results with the exact functions. The close agreement between the two results confirms that the proposed finite difference schemes are effective for the inverse determination of thermal conductivities.

The special feature of these approaches is that no prior information is required on the functional form of the unknown quantity. The numerical procedures are classified as first-order accurate methods. Higher-order approximations may be needed to obtain more accurate results. Although the algorithms are first-order accurate, they are useful and attractive for heat transfer inverse analysis due to their simplicity, stability, and high speed in numerical calculations. The techniques are applicable to linear and nonlinear spatially as well as temperature-dependent thermal conductivities. Although the present algorithms are developed for the inverse analysis of one-dimensional heat transfer, the procedures may also be extended to two-dimensional problems.

References

[1]Bayazitoglu, Y., and Suryanarayana, P. V. R., "Electromagnetic Levitation with Acoustic Modulation for Property Measurement," *Journal of Thermophysics and Heat Transfer*, Vol. 3, No. 3, 1989, pp. 351–353.

[2]Bayazitoglu, Y., Suryanarayana, P. V. R., and Sathuvalli, U. B., "A Thermal Diffusivity Determination Procedure for Solids and Liquids," *Journal of Thermophysics and Heat Transfer*, Vol. 4, No. 4, 1990, pp. 462–469.

[3]Murphy, J., and Bayazitoglu, Y., "Laser Flash Thermal Diffusivity Determination Procedure for High-Temperature Liquid Metals," *Numerical Heat Transfer, Part A*, Vol. 22, No. 1, 1992, pp. 109–120.

[4]Beck, J. V., "The Optimum Analytical Design of Transient Experiments for Simultaneous Determinations of Thermal Conductivity and Specific Heat," Ph.D. Dissertation, Michigan State Univ., East Lansing, MI, 1964.

[5]Nagler, R. G., "Transient Techniques for Determining the Thermal Conductivity of Homogeneous Polymeric Materials at Elevated Temperatures," *Journal of Applied Polymer Science*, Vol. 9, 1965, pp. 801–819.

[6]Pfahl, R. C., Jr., "Nonlinear Least Squares: A Method for Simultaneous Thermal Property Determination in Ablating Polymeric Materials," *Journal of Applied Polymer Science*, Vol. 10, 1966, pp. 1111–1119.

[7]Artyukhin, E. A., and Nenarokomov, A. V., "Coefficient Inverse Heat-Conduction Problem," *Journal of Engineering Physics*, Vol. 53, 1987, pp. 1085–1090.

[8]Artyukhin, E. A., Budnik, S. A., and Okhapkin, A. S., "Temperature-Measurement Optimization and Numerical Inverse Conduction-Treatment Solution," *Journal of Engineering Physics*, Vol. 55, No. 2, 1988, pp. 924–929.

[9]Beck, J. V., and Arnold, K. J., *Parameter Estimation in Engineering and Science*, Wiley, New York, 1977.

[10]Tikhonov, A. N., and Arsenin, V. Y., *Solution of Illposed Problems*, translation edited by F. John, Winston/Wiley, Washington, DC, 1977.

[11]Hensel, E., *Inverse Theory and Applications for Engineers*, Prentice–Hall, Englewood Cliffs, NJ, 1991.

[12]Jarny, Y., Delaunay, D., and Bransier, J., "Identification of Nonlinear Thermal Properties by an Output Least Square Method," *Proceedings of the 8th International Heat Transfer Conference* (San Francisco, CA), Vol. 4, Hemisphere, Washington, DC, 1986, pp. 1811–1816.

[13]Colonius, F., and Kunisch, K., "Stability for Parameter Estimation in Two Point Boundary Value Problems," *Journal für die Reine und Angewandte Mathematik*, Vol. 370, 1986, pp. 1–29.

[14]Richter, G. R., "An Inverse Problem for the Steady State Diffusion Equation," *SIAM Journal on Applied Mathematics*, Vol. 41, No. 2, 1981, pp. 210–221.

[15]Kubo, K., Mizuuchi, K., Ohnaka, I., and Fukusako, T., "Measurement of Thermal Properties of Sand Molds by Pouring Method," 50th International Foundry Congress, Cairo, 1983.

[16]Lund, J., and Vogel, C. R., "A Fully-Galerkin Method for the Numerical Solution of an Inverse Problem in a Parabolic Partial Differential Equation," *Inverse Problem*, Vol. 6, No. 2, 1990, pp. 205–217.

[17]Jarny, Y., Ozisik, M. N., and Bardon, J. P., "A General Optimization Method Using Adjoint Equation for Solving Multidimensional Inverse Heat Conduction," *International Journal of Heat and Mass Transfer*, Vol. 34, No. 11, 1991, pp. 2911–2919.

[18]Artyukhin, E. A., "Recovery of the Temperature Dependence of the Thermal Conductivity Coefficient from the Solution of the Inverse Problem," *High Temperature*, Vol. 19, 1981, pp. 698–702.

[19]Alifanov, O. M., and Mikhailov, V. V., "Solution of the Nonlinear Inverse Thermal Conductivity Problem by the Iteration Method," *Journal of Engineering Physics*, Vol. 35, No. 6, 1978, pp. 1501–1506.

[20]Tervola, P., "A Method to Determine the Thermal Conductivity from Measured Temperature Profiles," *International Journal of Heat and Mass Transfer*, Vol. 32, No. 8, 1989, pp. 1425–1430.

[21]Davidon, W. C., "Variable Metric Method for Minimization," Atomic Energy Commission Research and Development, Argonne National Lab., Rept. ANL-599, Univ. of Chicago, Chicago, IL, 1959.

[22]Fletcher, R., and Powell, M. J. D., "A Rapidly Convergent Descent Method for Minimization," *Computer Journal*, Vol. 6, No. 2, 1963, pp. 163–168.

[23]Ouyang, T., "Analysis of Parameter Estimation Heat Conduction Problems with Phase Change Using the Finite Element Method," *International Journal for Numerical Methods in Engineering*, Vol. 33, No. 10, 1992, pp. 2015–2037.

[24]Gear, C. W., *Numerical Initial Value Problems in Ordinary Differential*, Prentice–Hall, Englewood Cliffs, NJ, 1971.

[25]Silva, A. J., and Ozisik, M. N., "Inverse Problem of Simultaneously Estimating the Timewise Varying Strength of Two-Plane Heat Sources," *Journal of Applied Physics*, Vol. 73, No. 5, 1993, pp. 2132–2137.